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LONGITUDINAL VIBRATIONS OF ELASTIC RODS OF STEPWISE-VARIABLE CROSS-SECTION COLLIDING WITH A RIGID OBSTACLE†

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A frequency method of solving the problem of the longitudinal vibrations of elastic rods of stepwise-variable cross-section is proposed, taking into account or ignoring the energy dissipation when they collide with a rigid obstacle. A Laplace transformation is applied to the equation of longitudinal vibrations of the rod when there are non-zero initial conditions. For the inhomogeneous differential equation obtained, the boundary-value problem of finding the Laplace-transformed longitudinal boundary forces as functions of the boundary displacements is solved. The equations of equilibrium of the junction points, which are a system of equations for the unknown junction displacements, are then set up. Since the corresponding coefficients are obtained by exact integration, there is no constraint on the length of the rod sections. An inverse transformation is carried out by using extremal points of the amplitude-phase frequency characteristics [1] or by direct integration. A rod of constant cross-section of finite length is considered as a test example. The result is compared with the well-known wave solution [2]. The proposed approach is described here for the first time and imposes practically no constraints on the class of problems that can be considered, whereas the existing approach leads to unsurmountable difficulties when there are several sections of the rod. © 2001 Elsevier Science Ltd. All rights reserved.

1. A ROD OF CONSTANT CROSS-SECTION

Consider a continuous elastic rod of length l, which collides with velocity V_0 with a rigid obstacle. We will determine the displacement of points of the rod after the collision. We will assume that contact is maintained between the obstacle and the rod after the collision, i.e. the rod does not rebound. If the constraint is unilateral, the problem can be regarded as a piecewise-linear problem. The criterion which indicates a change to a different version of the solution is a change in the sign of the velocity at the point of contact.

The equation of longitudinal vibrations of the rod, taking the internal resistance forces into account, has the form

$$\mu u_{tt} - EFu_{xx} - EF\gamma u_{xxt} = 0 \tag{1.1}$$

where u is the displacement of the points of the rod, μ is the mass per unit length of the rod, E is the modulus of elasticity, F is the cross-section are, and γ is the coefficient of resistance.

We will specify the following boundary and initial conditions

$$u|_{x=0} = 0, \quad u_x|_{x=l} = 0 \tag{1.2}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = -V_0 \tag{1.3}$$

We carry out a Laplace transformation of Eq. (1.1) and boundary conditions (1.2) with the specified initial conditions (1.3). Equation (1.1) and boundary conditions (1.2) can then be written as follows:

$$\mu p^2 U - EFU_{xx}(1+\gamma p) = -\mu V_0, \tag{1.4}$$
$$U|_{x=0} = 0, \quad U_x|_{x=l} = 0$$

where p is the Laplace transformation parameter and $U = U(p)$ is the Laplace transform of the displacement of points of the rod.

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The solution of Eq. (1.4), ignoring the energy dissipation (i.e. why $\gamma = 0$), was obtained previously in [2], and its original was found. The solution of this problem can be obtained by other methods [3]. The solution of Eq. (1.4) can be written in the following form (everywhere henceforth summation is carried out from $n = 1$ to $n = \infty$)

$$U(\alpha) = \sum\left[\left(T_{n2}^2 + T_{n1}p + 1\right)\|u_n\|^2\right]^{-1}\lambda_n u_n(\alpha)\int_V f^T u_n dV \tag{1.5}$$

$$u_n = \sin(\Pi_n x),\quad \omega_n = \Pi_n\left(\frac{E}{\rho}\right)^{1/2},\quad \Pi_n = \frac{2n-1}{2l}\pi$$

$$f^T = -\rho V_0,\quad \lambda_n = \frac{1}{\omega_n^2},\quad \lambda_n = T_{n2}^2,\quad \rho = \frac{\mu}{F}$$

where u_n are the natural oscillation modes, ω_n are the natural frequencies, and ρ is the density of the rod.

As it applies to this problem, we have

$$T_{n1} = \gamma,\quad \lambda_n = \frac{\rho}{\Pi_n^2 E} = T_{n2}^2$$

$$\int_V f^T u_n dV = -\int_0^l \rho V_0 \sin(\Pi_n x)dx = -\frac{\rho V_0}{\Pi_n}$$

$$\|u_n\|^2 = \int_0^l \rho u_n^2 dx = \rho\int_0^l \sin^2(\Pi_n x)dx = \frac{\rho l}{2}$$

Then, we obtain for the original

$$u(x,t) = \sum\frac{-8V_0 l\sin(\Pi_n x)\sin\left((E/\rho)^{1/2}\Pi_n t\right)}{(E/\rho)^{1/2}\pi^2(2n-1)^2} \tag{1.6}$$

Solution (1.6) describes wave phenomena in the rod with sufficient accuracy while retaining five oscillation modes. Solution (1.6) was compared with the exact solution, obtained previously in [2], up to 50 oscillation modes. The accuracy amounts to 0.005%. In Fig. 1 we show curves of the solution (1.6) as a function of the number of terms of the series: curve 1 is the test curve [2], and curves 2, 3 and 4 are curves drawn using formula (1.6) for $n = 7$, 5 and 3 respectively ($l = 1.595\ m$, $\rho = 7820$ kg/m^3, $\gamma = 0.01$, $F = 0.025$ m^2 and $E = 2.1 \times 10^5$ MPa). The disagreement for $n = 7$ does not exceed 2.8%.

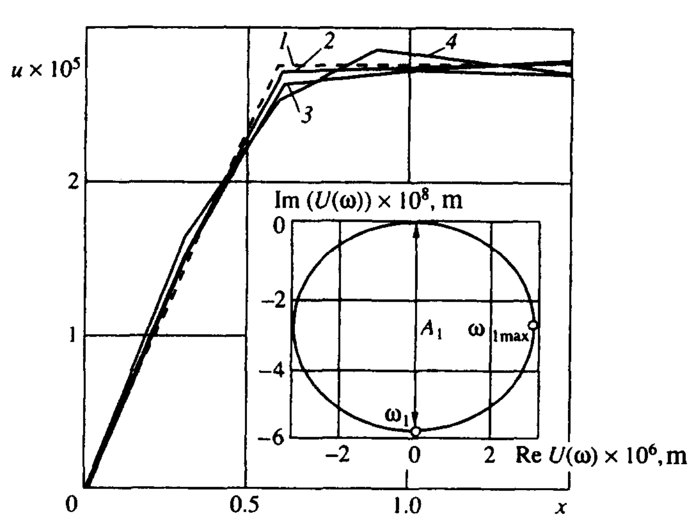

Fig. 1

We will propose a frequency method of solving the problem which we will compare with the known solution in the form of a series in the oscillation modes.

We will introduce the dimensionless variable $\xi = x/l$ and consider the homogeneous equation of longitudinal vibrations of the rod, taking energy dissipation into account

$$U_{\xi\xi} + aU = 0, \quad a = -\frac{\mu}{FE(1+p\gamma)} p^2 l^2 \tag{1.7}$$

The solution of Eq. (1.7) has the form

$$U = c_1 \cos\alpha\xi + c_2 \sin\alpha\xi, \quad \alpha = a^{1/2}$$

where c_1 and c_2 are integration constants. The following transfer matrix corresponds to this solution [3]

$$K = \left\| \begin{matrix} \cos\alpha\xi & \alpha^{-1}\sin\alpha\xi \\ -\alpha\sin\alpha\xi & \cos\alpha\xi \end{matrix} \right\| \tag{1.8}$$

Using transfer matrix (1.8), we can set up relations of the method of initial parameters for the auxiliary problem with a perturbing harmonic force $\mu V_0 \sin\omega t$

$$V(\xi) = K(\xi) r_0 + l \int_0^l K(\xi - s) r(s) ds \tag{1.9}$$

where $r_0^T = (U_0, N_0)$ is the vector of the initial parameters and $r(s) = (0, -\mu V_0)$ is the vector of the distributed harmonic force. Using the first row of matrix relation (1.9), we solve the boundary-value problem, which consists of finding the amplitudes of the boundary forces N_j and N_k in terms of the amplitudes of the forced harmonic displacements of the ends of the rod U_j and U_k. We obtain

$$\begin{aligned} N_{jk} &= S_{jk} U_j - T_{jk} U_k + T_{jk}[u_k] \\ N_{kj} &= S_{kj} U_k - T_{kj} U_j + T_{kj}[u_j] \end{aligned} \tag{1.10}$$

Here

$$S_{jk} = S_{kj} = \frac{E_{kj} F_{kj} (1 + i\omega\gamma_{kj}) \alpha_{kj}}{l_{kj}} \frac{\cos\alpha_{kj}}{\sin\alpha_{kj}}$$

$$T_{jk} = T_{kj} = \frac{E_{kj} F_{kj} (1 + i\omega\gamma_{kj})}{l_{kj}} \frac{\alpha_{kj}}{\sin\alpha_{kj}}$$

$$[u_j] = [u_k] = -\frac{\mu_{kj} V_0 l_{kj}^2}{E_{kj} F_{kj} (1 + i\omega\gamma_{kj}) \alpha_{kj}} \int_0^1 \sin(\alpha_{kj} s) ds = -\frac{\mu_{kj} V_0 l_{kj}^2}{E_{kj} F_{kj} (1 + i\omega\gamma_{kj})} \frac{1 - \cos\alpha_{kj}}{\alpha_{kj}^2}$$

The subscripts j and k indicate the beginning and end of the rod respectively, and $[u_j]$ and $[u_k]$ are quantities which take into account the effect of the local load and the initial conditions.

From the second equation of (1.10) we obtain an expression for the Laplace transform of the displacement of the end of the rod for $p = i\omega$

$$U_k(\omega) = -\frac{\mu_{kj} V_0 l_{kj}^2 (1 - \cos\alpha_{kj})}{\alpha_{kj}^2 E_{kj} F_{kj} \cos\alpha_{kj} (1 + i\omega\gamma_{kj})} \tag{1.11}$$

On the lower right in Fig. 1 we show a graph of the amplitude-phase frequency characteristic of the displacements of the end of the rod, where A_1 is the amplitude of the loop, ω_1 is the frequency corresponding to the maximum imaginary component of the loop, and $\omega_{1\max}$ is the frequency corresponding to the maximum real component of the loop. We have established that the amplitude-phase frequency characteristic constructed using (1.5), for $x = 1$, agrees completely with the graph obtained using (1.11), which gives the expression for the sum of the series (with $\gamma = 0$)

$$\sum \frac{4V_0 \sin(\Pi_n l_{kj})}{\rho\pi(2n-1)} \frac{1}{-\omega^2 + \omega_n^2} = \frac{\mu_{kj} V_0 l_{kj}^2 (1 - \cos\alpha_{kj})}{\alpha_{kj}^2 E_{kj} F_{kj} \cos\alpha_{kj}}$$

It should be noted that, unlike solution (1.5), the proposed approach does not require a knowledge of the natural frequencies and oscillation modes, which considerably simplifies the solution algorithm. The problem of the collision of a point mass with a rod was solved previously by a similar method in [4]. The main difference with the problem considered here is that the amplitude-phase frequency characteristic for the fictitious distributed load $-\mu V_0 \sin\omega t$, determined by the velocity of the collision between the rod and the obstacle, is constructed as an auxiliary problem.

Transform (1.5) has the tabulated original

$$u(x,t) = \sum_{n=1}^{m} \frac{2k_n}{\left(4T_{n2}^2 - T_{n1}^2\right)^{1/2}} \exp\left(-\frac{T_{n1}}{2T_{n2}^2}\right) \sin\left(\frac{\left(4T_{n2}^2 - T_{n1}^2\right)^{1/2}}{2T_{n2}^2} t\right) \tag{1.12}$$

where $u(x, t)$ is the impulse transfer function by means of which the transition process occurs and m is the number of loops which appear on the amplitude-phase frequency characteristic.

It is well-known that one term of the series in solution (1.5) corresponds to each loop of the amplitude-phase frequency characteristic. There is a unique relation between the extremum points of the amplitude-phase frequency characteristic and the coefficients of the corresponding terms of the series in solution (1.5), which is used in this paper to carry out the inverse Laplace transformation. To do this, it is necessary to determine the characteristic frequencies and amplitudes from the amplitude-phase frequency characteristic. In this case [1]

$$k_n = A_n T_{n1} \omega_n, \quad \frac{T_{n1}}{T_{n2}} = 1 - \frac{\omega_{n\max}^2}{\omega_n^2}, \quad T_{n2} = \frac{1}{\omega_n}$$

where A_n is the amplitude of the n-th loop, ω_n is the frequency corresponding to the maximum imaginary component of the n-th loop, $\omega_{n\max}$ is the frequency corresponding to the maximum real component of the n-th loop and k_n is the amplification factor.

The same result can be obtained by numerical integration with $t = 0, 1, \ldots,$

$$u(x,t) = \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}\left(U(\omega) e^{i\omega t}\right) d\omega$$

2. A ROD OF STEPWISE-VARIABLE CROSS-SECTION

Dividing the rod into parts, along which the physical and geometrical parameters are constant, we set up the equations of dynamic equilibrium of the junction points, which are a system of equations for the unknown junction displacements [3]

$$\begin{gathered} -T_{k-2,k} U_{k-2} + \left(S_{k-2,k} + S_{k,k+2} + C_{k,k+1}\left(1 + \gamma_{k,k+1} p\right) + C_k(1 + \gamma_k p) + p^2 m_k\right) U_k - \\ - C_{k,k+1} U_{k+1} - T_{k,k+2} U_{k+2} = -T_{k-2,k}\left[u_{k-2}\right] - T_{k,k+2}\left[u_{k+2}\right] + m_k p \dot{U}_{k,0} \\ m_{k+1} p^2 U_{k+1} + C_{k,k+1}\left(1 + \gamma_{k,k+1} p\right)\left(U_{k+1} - U_k\right) = m_{k+1} \dot{U}_{k+1,0} \\ k = 1, 2, \ldots, n \end{gathered} \tag{2.1}$$

where $\dot{U}_{k,0}$ and $\dot{U}_{k+1,0}$ are the initial velocities of the junction points k and $k+1$, C_k is the concentrated stiffness and m_k is the mass of the junction point.

The theoretical scheme of the rod is shown in Fig. 2.

Solving the system obtained with $p = i\omega$, we construct the amplitude-phase frequency characteristics of the chosen cross-sections of the rod. These amplitude-phase frequency characteristics can be regarded

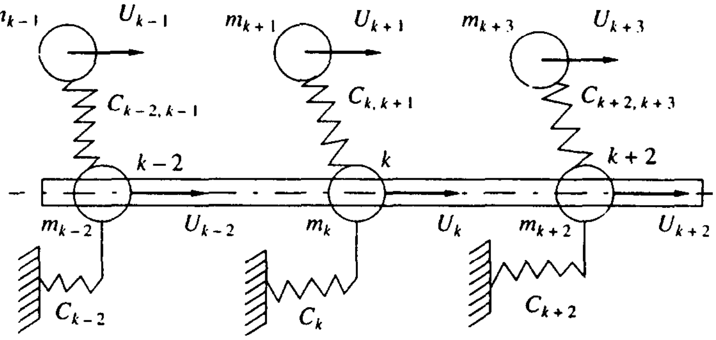

Fig. 2

as a graphical form of a unilateral Fourier transformation. Since all the singular points of the corresponding expressions lie to the left of the imaginary axis, the inverse transformation can be carried out assuming $p = i\omega$, i.e. using the amplitude-phase frequency characteristics. The problem of constructing the amplitude-phase frequency characteristics where the field of the initial velocities multiplied by the mass per unit length of the rod is taken as the force, is an auxiliary problem. Usually when constructing the amplitude-phase frequency characteristics one takes the perturbing forces as the input action, and the inverse Laplace transformation is carried out by numerical integration or some other method. Since the corresponding coefficients are obtained by accurate integration, the rod sections can be of any length.

As a technical example we will consider the dynamic phenomena which arise in the incident parts of a forging hammer when it collides rigidly with a lower block. A theoretical representation of this system is shown in Fig. 3 (A is a shaft, B is the head of the hammer and C is the upper block). The following system of resolvents corresponds to this scheme

$$\begin{aligned} &S_{1,2}U_1 - T_{1,2}U_2 = -T_{1,2}[u_2] \\ &-T_{1,2}U_1 + (S_{1,2} + S_{2,3})U_2 - T_{2,3}U_3 = -T_{1,2}[u_1] - T_{2,3}[u_3] \\ &-T_{2,3}U_2 + (S_{2,3} + S_{3,4})U_3 = -T_{2,3}[u_2] - T_{3,4}[u_4] \end{aligned} \tag{2.2}$$

from which we find the displacements in the chosen sections.

To check this method we divide a rod of constant cross-section into three parts of length $l_{1,2}$, $l_{2,3}$ and $l_{3,4}$. It is found that the graphs of the amplitude-frequency phase characteristics constructed with the

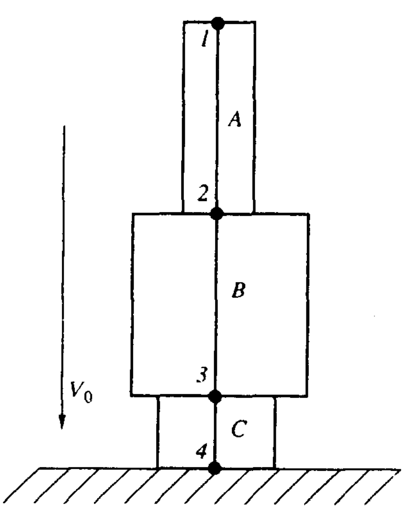

Fig. 3

same initial data from formula (1.5) with $x = l_{3,4}$, $x = l_{3,4}, + l_{2,3}$ and $x = l$, agree completely with the graphs of $U_1(\omega)$, $U_2(\omega)$ and $U_3(\omega)$ respectively, obtained from system (2.2). Knowing the displacements of the beginning and end of a part, from formulae (1.10) we can calculate the longitudinal forces $N(\omega)$, the originals of which are related to the stresses $\sigma(t)$ and strains $\varepsilon(t)$ as follows:

$$\sigma(t) = N(t)/F, \quad \varepsilon(t) = N(t)/(EF)$$

Thus, we have developed a method of calculating the longitudinal vibrations of a rod of piecewise-variable cross-section when it collides with a rigid obstacle. The approach described enables us to solve problems of the dynamics of rods of piecewise-variable cross-section when there is an unlimited number of elastically connected masses, for an arbitrary force applied at the ends and along the length of the rod, i.e. when direct treatment of the corresponding formulae is not feasible.

To compare the accuracy of the results we carried out experimental investigations on a model M1345 forging hammer produced by the "Aviastar" Company (Ul'yanovsk). The initial data are as follows:

$$E = 2.1 \times 10^5 \text{ MPa}, \quad \rho = 7850 \text{ kg/m}^3, \quad V_0 = 0.7 \text{ m/s}$$

Beginning of the part	1	2	3
End of the part	2	3	4
l, m	1.595	0.905	0.3
F, m^2	0.025	0.39	0.204

The calculation was carried out in the following sequence. We initially found the displacements and stresses in sections 1, 2 and 3. Then, from (1.9), we calculated the displacements along the length of a part of the rod. Calculations were made at nine junction points of the system on a computer. In Fig. 4 we show the theoretical stresses (the continuous curve) and the experimental stresses (the dashed curve) at the chosen junction points (n is the number of the junction point) at the instant of time $t = 0.02$ s.

It can be seen from the results that the dangerous sections, where fractures of the structure occur, are the junction of the shaft and the hammer head (junction point 4) and, to a lesser extent, the junction with the upper block (junction point 9). This is confirmed experimentally. The disagreement between the theoretical and experimental results amounted to no more than 18%. If this problem were to be solved by the method of finite elements, the system of resolvents would be of the order of 100–130 equations.

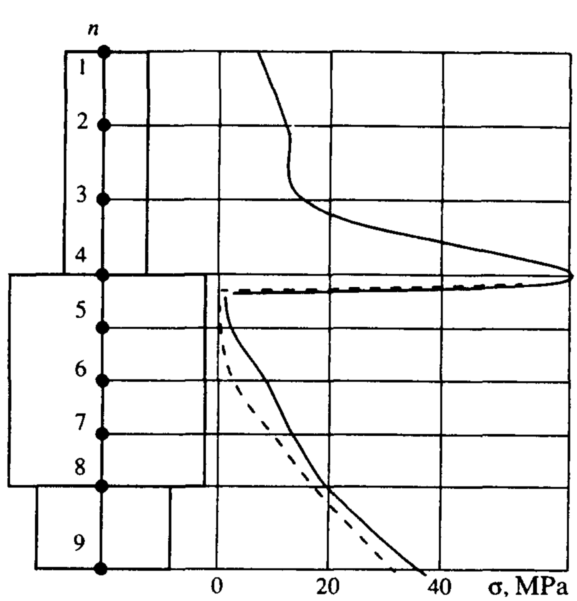

Fig. 4

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